Antidirected paths in 5-chromatic digraphs

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Abstract. Let $T_5$ be the regular 5-tournament. B. Grünbaum proved that $T_5$ is the only 5-tournament which contains no copy of the antidirected path $P_4$. In this paper, we prove that except for $T_5$, any connected 5-chromatic oriented digraph in which each vertex has out-degree at least two contains a copy of $P_4$. It will be shown, by an example, that the condition that each vertex have out-degree at least two is indispensable.

1. Introduction

The digraphs considered here have no loops or multiple edges. An oriented graph is a digraph in which, for every two vertices $x$ and $y$, at most one of $(x, y)$, $(y, x)$ is an edge. The digraphs used in this paper are all oriented graphs. By $G(D)$ we denote the underlying graph of a digraph $D$. The chromatic number of a digraph is the chromatic number of its underlying graph. A graph $G$ is $k$-critical if $\chi(G) = k$ and $\chi(G - v) = k - 1$ for any vertex $v$ in $V(G)$.

A block of an oriented path is a maximal directed subpath. We recall that the length of a path is the number of its edges. The antidirected path is an oriented path in which each block is of length 1. $P_n$ denotes the antidirected path of length $n$, beginning by a backward edge.

The problem of determining which oriented paths lie in a given $n$-chromatic digraph $D$ is a well-known problem. When $D$ is an $n$-tournament, the problem has been completely resolved (Havet and Thomassé [6]). However, the case of an arbitrary $n$-chromatic digraph is still an open question. We know only that an $n$-chromatic digraph contains a directed path of length $n - 1$ (Roy [7], Gallai [4]), and a path of length $n - 1$ formed by two blocks, one of which has length 1 [2]. In this paper, we will be interested in the antidirected paths. In order to generalize the results found on tournaments to arbitrary digraphs, and as a first step in this direction, we generalize to 5-chromatic digraphs a particular result of Grünbaum on 5-tournaments: any 5-tournament except for the regular tournament $T_5$ contains a copy of $P_4$.

2. The main result

Theorem 1. Let $D$ be a 5-chromatic connected digraph, distinct from $T_5$, in which each vertex has out-degree at least two. Then $D$ contains a copy of $P_4$.

To prove this theorem, we need several lemmas.
Lemma 1 (Grünbaum [5]). Except for $T_5$, any 5-tournament contains a copy of $P_4$.

Corollary 1. Let $D$ be as in the above theorem. If $D$ contains $T_5$, then $D$ contains a copy of $P_4$.

Proof. We first observe that:
1) each vertex of $T_5$ in an end of a copy of $P_2$ and a copy of $P_3$;
2) for any two vertices of $T_5$, there is a copy of $P_2$ in $T_5$ having one of these vertices as an end and not containing the other.

Since $D$ is different from $T_5$, there exists an edge $xy$ in $G(D)$ such that $x \notin V(T_5)$ and $y \in V(T_5)$. If $(x, y) \in E(D)$ it forms together with a path $P_4$ in $T_5$ having $y$ as an end, a path $P_4$. Otherwise, since $d^+(x) \geq 2$, there is a vertex $x'$ of $D$ distinct from $y$ such that $x \rightarrow x'$. We choose a path $P_2$ in $T_5$ having either $y$ or $x'$ as an end and not containing the other; such a path exists by (1) and (2). Together with the path $yxx'$, it forms a copy of $P_4$. \hfill \Box

In the sequel, $D$ will denote an oriented digraph as described in theorem 1; by the above corollary we may assume that $D$ contains no 5-tournament as a subdigraph. Moreover, we suppose to the contrary that $D$ contains no copy of $P_4$. Let $D'$ be a 5-critical subdigraph of $D$ and let $D^o$ be the subdigraph of $D'$ induced by the vertices of out-degree at least three in $D'$.

Let $G$ be a graph which contains no $K_{2n+1}$, where $n \geq 2$. Suppose that we can orient $G$ in such a way that each vertex has in-degree at most $n$. It is shown in [?] that $\chi(G) \leq 2n$. We have then the following lemma

Lemma 2. $D^o$ is not empty.

Proof. Otherwise, we have $d^+_D(x) \leq 2$ for every $x$ in $D'$. Since $D'$ contains no 5-tournament, $\chi(D') \leq 4$, a contradiction. \hfill \Box

Lemma 3. Let $v$ be a vertex of $D$ and let $x$, $y$ be two vertices in $N^-(v)$. If $x \in V(D')$, then $y \notin V(D')$.

Proof. Suppose that $y \in V(D')$. The set $(N^+(x) \cup N^+(y)) \setminus \{v, x, y\}$ contains two distinct vertices $x'$ and $y'$ such that $x \rightarrow x', y \rightarrow y'$; the path $x'xvyy'$ is a copy of $P_4$, a contradiction. \hfill \Box

Corollary 2. For every vertex $v$ in $D^o$, $d^-_{D^o}(v) \leq 1$.

Lemma 4. Let $H$ be a connected digraph in which each vertex has in-degree at most one. Then $H$ contains at most one cycle.

Lemma 5. Let $v$ be a vertex of $D$ such that $d^+(v) \geq 3$ and let $x$, $y$ and $z$ be three distinct vertices in $N^+(v)$. Suppose that $x \rightarrow y$. Then:
1) $x \rightarrow z$.
2) $yz \notin E(G(D))$.
3) $N^-(y) = N^-(z) = \{v, x\}$.

Proof. 1) Since $d^+(x) \geq 2$, there is a vertex $x'$ in $D$ distinct from $y$ such that $x \rightarrow x'$. Then $x' = z$, since otherwise the path $x'xyyz$ is a copy of $P_4$.
2) Suppose that $y \rightarrow z$. We have, by (1), $y \rightarrow x$; but $x \rightarrow y$, a contradiction. We use the same argument if $z \rightarrow y$.
3) Let $H = D[v, x, y, z]$. We first remark that if $u \in \{y, z\}$ and $u'$ is any vertex
of $D$ distinct from $u$, there exists a path $P_2$ in $H$ having $u$ as an end and not containing $u'$. Suppose, to the contrary, that $N^-(u) \setminus \{v, x\}$ is not empty, where $u \in \{v, z\}$. Let $w \in N^-(u) \setminus \{v, x\}$ and let $u'$ be a vertex of $D$ distinct from $u$ such that $w \rightarrow u'$. By the above remark, there exists a path $P_2$ in $H$ having $u$ as an end and containing no $u'$. A copy of $P_4$ is then formed by the path $u'wu$ and $P_2$, a contradiction.

\[\square\]

**Corollary 3.** Let $x$ and $y$ be two adjacent vertices of $D$. Suppose that there exist two vertices $v$ and $v'$ of $D$ such that $\{x, y\} \subseteq N^+(v) \cap N^-(v')$. Then $N^+(v) = \{x, y\}$.

**Proof.** Suppose that $\{x, y\} \subset N^+(v)$ and let $z \in N^+(v) \setminus \{x, y\}$; we may suppose that $x \rightarrow y$, then we have, by the above lemma, $x \rightarrow z$ and $yz \notin E(G(D))$, so $\{y, z, v'\} \subseteq N^+(x)$. Since $y \rightarrow v'$ we have $y \rightarrow z$, a contradiction.

\[\square\]

**Lemma 6.** $D'$ is an independent set of $D$.

**Claim 1.** Any connected component $L$ of $D'$ contains a vertex $v$ such that $N^+(v) \cap (V(D') \setminus V(D'))$ has at least two vertices.

**Proof.** If $L$ is a cycle, then each vertex of $L$ satisfies the claim; otherwise $L$ contains a vertex $v$ of out-degree zero in $D'$, and so $N^+(v) \subseteq V(D') \setminus V(D')$.

\[\square\]

**Proof of lemma 6:** Suppose to the contrary that $D'$ is not an independent set, then there is a connected component $L$ of $D'$ containing at least two vertices. We can choose a vertex $v$ in $L$ satisfying the claim such that $d_L^-(v) = 1$. Let $v'$ be a vertex in $L$ such that $v' \rightarrow v$ and let $v_1$, $v_2$ and $v_3$ be three vertices in $N_{D'}^+(v)$ such that $\{v_1, v_2\} \subseteq V(D') \setminus V(D')$. The digraph $D'$ is 5-critical, so any vertex has degree at least 4 in $D'$. Since for any $i \in \{1, 2\}$, $d_{D'}^-(v_i) \leq 2$, we have $d_{D'}^-(v_i) \geq 2$. Therefore, there is a vertex $u$ of $D'$ and $j \in \{1, 2\}$ such that $u \notin \{v, v_1, v_2\}$ and $u \rightarrow v_j$; we have either $u \notin \{v, v_1, v_2, v_3\}$ or $u = v_3$. In the latter case $v_3 \notin V(D')$ by lemma 3. We have $d_{D'}(v_3) \geq 2$, so there is a vertex $w$ of $D'$ such that $w \notin \{v, v_1, v_2, v_3\}$ and $w \rightarrow v_3$, thus we may assert that there exists a vertex $u$ of $D'$ and $j \in \{1, 2, 3\}$ such that $u \notin \{v, v_1, v_2, v_3\}$, $v_j \notin D'$ and $u \rightarrow v_j$. Let $u'$ be a vertex of $D$ distinct from $v_j$ such that $u \rightarrow u'$. If $u' \neq v$, the path $u'v_jv_k$ is a copy of $P_4$, where $h \in \{1, 2, 3\} \setminus \{j\}$ is chosen such that $u' \neq v_h$ , a contradiction. Otherwise let $w$ be a vertex in $N^+(v_j) \setminus \{v, v_j, u\}$. Such a vertex exists since $d^+(v_j) \geq 3$ and $v_j \notin N^+(v')$ by lemma 3. The path $v_jvu'w$ is a copy of $P_4$, a contradiction.

\[\square\]

**Corollary 4.** Let $v$ be a vertex in $D'$. Then
1) there exist two vertices $x$, $y$ in $N_{D'}^+(v)$ such that $x \rightarrow y$;
2) $d_{D'}^+(v) = 3$.

**Proof.** 1) If (1) does not hold, let $v_1$, $v_2$ and $v_3$ be three vertices in $N_{D'}^+(v)$. For each $i \in \{1, 2, 3\}$, there is a vertex $u_i$ in $D'$ such that $u_i \notin \{v, v_1, v_2, v_3\}$ and $u_i \rightarrow v_i$; the case $u_1 = u_2 = u_3$ does not occur by lemma 3. Suppose that $u_1 \neq u_2$. Then $u_1 \rightarrow v$, $i = 1, 2$, since otherwise $D'$ contains a path $P_4$. But now the path $v_1u_1v_2v_3$ is a copy of $P_4$.

2) Suppose, to the contrary, that $d_{D'}^+(v) \geq 4$ and let $x$, $y$ be two vertices in $N_{D'}^+(v)$ such that $x \rightarrow y$. By lemma 5, $x \rightarrow z$ for every $z \notin x$ in $N_{D'}^+(v)$.  \[\square\]
Thus $d^+_{D^o}(v) \geq 3$ and so $x, v$ belong to the same connected component of $D^o$, a contradiction.

In the sequel, we will need the following theorem proved by Gallai [3].

**Theorem 2.** Let $G$ be a $k$-critical graph, where $k$ is a positive integer. Let $G_m$ be the subgraph of $G$ induced by the vertices of degree $k-1$. Then each block of $G_m$ is either complete or a chordless odd cycle.

$D_4$ will denote the subdigraph of $D'$ induced by the vertices of degree 4.

**Lemma 7.** Any vertex of $D'$ has in-degree (in $D'$) at least 2.

**Proof.** It is clear that any vertex in $V(D') \setminus V(D^o)$ has in-degree at least 2 in $D'$. Let $v \in V(D^o)$ and $N^+_D(v) = \{x, y, z\}$ where $x \rightarrow y$ and $x \rightarrow z$. By lemma 5, we have $yz \notin E(G(D))$ and $N^-(y) = N^-(z) = \{v, x\}$, so $d_D(y) = d_D(z) = 4$. For every $u$ in $D'$ we have $u \rightarrow v$ whenever $u \rightarrow x$, since otherwise we have a path $P_4$ in $D$; consequently, if $d_{D'}(v) = 1$ then $d_{D'}(x) = 2$ and so $d_{D'}(v) = d_{D'}(x) = 4$. Therefore $x, y, z$ and $v$ are in the same block of $D_m$, so $D'[v, x, y, z]$ is complete, which is a contradiction since $yz \notin E(G(D))$. □

We now associate to each vertex $v$ in $D^o$ the set $S(v) = \{t(v), t'(v), v_0, ..., v_g(v), v_{g(v)+1}\}$, $0 \leq g(v) \leq 5$, defined as follows (see Figure 1)

\[ \{v_0, t(v), t'(v)\} = N^+_D(v) \] where $v_0 \rightarrow t(v)$ and $v_0 \rightarrow t'(v)$, $v_1 = v$. Set $T(v) = \{t(v), t'(v)\}$. If $d_{D'}(v_0) \geq 3$, put $g(v) = 0$; if not, let $v_2$ be the unique vertex of $D'$ distinct from $v_1$ such that $v_2 \rightarrow v_0$. We have $v_2 \rightarrow v_1$. Again, if $d_{D'}(v_1) \geq 3$, put $g(v) = 1$; otherwise, let $v_3$ be the unique vertex of $D'$ distinct from $v_2$ such that $v_3 \rightarrow v_1$; such a vertex exists by lemma 7. We have $v_2 \rightarrow v_1$, since otherwise we have either a path $P_4$ in $D$ or $d_{D'}(v_0) \geq 3$. We may continue this process until meeting the first vertex of in-degree at least three in $D'$; call this vertex $v_{g(v)}$, where $g(v)$ is the number of iterations required. Such a vertex exists and $g(v) \leq 5$.

In fact, suppose that $v_1, ..., v_5$ are defined as above and $d_{D'}(v_1) = 2$, $i = 1, ..., 4$. By corollary 3, we have $d_{D'}(v_i) = 2$, $i = 2, ..., 5$. If $d_{D'}(v_5) = 2$ the vertices $v_2, ..., v_5$ will be in the same block of $D_4$. By theorem 2, $D'[v_2, ..., v_5]$ is complete, which is a contradiction since $v_2v_5 \notin E(G(D))$.

Set $O(v) = \{z \in D' : z \neq v_{g(v)+1} \text{ and } z \rightarrow v_{g(v)}\}$; we have $z \rightarrow v_{g(v)+1}$ for every $z$ in $O(v)$.

**Lemma 8.** Let $u$ and $v$ be two distinct vertices of $D^o$. We have:
\( S(u) \cap S(v) = \phi. \)

**Proof.** We first remark that \( N^+_D(x) \subseteq S(v) \cup O(v) \) for all \( x \) in \( S(v) \) and \( N^-_D(x) \subseteq S(v) \) for all \( x \) in \((S(v) \cup O(v))\setminus\{v\}\). By lemma 5 and corollary 3 we have \( d^+_D(x) = 2 \) for all \( x \) in \((S(v) \cup O(v))\setminus\{v\}\), so \( u \notin S(v) \cup O(v) \). Let \( x \) be in \( N^+_D(x) \). If \( x \in S(v) \), then \( N^-_D(x) \subseteq S(v) \cup O(v) \), but \( u \in N^-_D(u_0) \), a contradiction. Moreover \( u_0 \notin O(v) \) since otherwise \( T(u) = N^+_D(u_0) \subseteq S(v) \). The same argument proves that \( \{t(v), t'(v), v_0, v\} \cap S(u) = \phi \).

Suppose that \( u_i \notin S(v) \cup O(v) \) for some \( i \), \( 0 \leq i \leq g(u) \); if \( u_{i+1} \in S(v) \cup O(v) \), then \( u_{i+1} \in S(v) \cup O(v) \), so \( u_{i+1} \in S(v) \cup O(v) \setminus \{v, v_0, t(v), t'(v)\} \). Thus \( N^+_D(u_{i+1}) \subseteq S(v) \); but \( u_i \in N^+_D(u_{i+1}) \), a contradiction.

**Lemma 9.** Set \( L = \{v_{g(v)} : v \in D^o\} \). We have:

(i) \( d^-_D(x) = 3 \) for any \( x \) in \( L \).

(ii) \( d^-_D(x) = 2 \) otherwise.

**Proof.** Let \( s \) and \( p \) be the numbers of vertices in \( D^o \) and \( D' \) respectively. We have:

\[
e(D') = \sum_{v \in V(D')} d^-_D(v) = \sum_{v \in L} d^-_D(v) + \sum_{x \in V(D') \setminus L} d^-_D(v)
\]

By lemmas 7 and 8:

\[(2.1) \quad e(D') \geq 3s + 2(p - s). \]

On the other hand:

\[
e(D') = \sum_{v \in V(D')} d^+_D(v) = \sum_{v \in V(D')} d^+_D(v) + \sum_{v \in V(D') \setminus V(D^o)} d^+_D(v)
\]

so

\[(2.2) \quad e(D') \leq 3s + 2(p - s). \]

If (i) or (ii) does not hold the inequality (2.1) will be strict, which contradicts inequality (2.2).

**Corollary 5.** For any vertex \( v \) in \( D' \), \( O(v) \) contains exactly two vertices.

**Proof of Theorem 1.** Define the sets:

\[
S = \{v \in V(D') : S(v)\}, \quad O = \{v \in V(D') : O(v)\}, \quad T = \{v \in V(D') : T(v)\}.
\]

We have \( |O| \leq |T| \). If \( O = T \), then \( N^+_D(v) \subseteq S \) for every \( v \) in \( S \). Since \( D' \) is critical, it must be connected and so \( D' = D'[S] \). We define a colouring \( c \) of \( D' \) as follows: Let \( v \) be a vertex in \( D^o \) Put \( c(t(v)) = c(t'(v)) = 1, c(v_0) = 2, c(v_1) = 3. \) If \( g(v) = 1 \), put \( c(v_2) = 4. \) If \( g(v) > 1 \), the colours 1, 2 and 3 suffice to colour \( S(v) \setminus \{v_{g(v)}, v_{g(v)+1}\} \). Put \( c(v_{g(v)}) = 4 \) and \( c(v_{g(v)+1}) = i \) where \( i \in \{2, 3\} \) is chosen such that \( i \neq c(v_{g(v)-1}) \). It is clear that \( c \) is a proper 4-colour of the 5-chromatic digraph \( D' \), a contradiction.

If \( O \neq T \) then, since \( |O| \leq |T| \), there is a vertex \( v \) in \( D^o \) such that either \( t(v) \notin O \) or \( t'(v) \notin O \). Suppose, without loss of generality, that \( t(v) \notin O \). Then \( N^+_D(t(v)) \cap S = \phi \). Let \( N^+_D(t(v)) = \{u, u'\} \). We have \( \{u, u'\} \cap (D^o \cup L) = \phi \), so \( d^-_D(u) = d^-_D(u') = d^-_D(v) = 2 \) and \( d^+_D(u) = d^+_D(u') = 4 \). On the other hand, there exists a vertex \( w \) in \( D' \) such that \( w \notin \{u, u'\} \) and \( N^+_D(w) \cap \{u, u'\} = \phi. \)
We have $N^+_D(w) = \{u, u'\}$ since $D'$ contains no path $P_4$ and $w t(v)$ cannot be an edge of $G(D')$; thus $d_{D'}(w) = 4$.

Since $d_{D'}(t(v)) = 4$, the vertices $t(v), u, u'$ and $w$ are in a block of $D_4$ which is neither complete nor a chordless odd cycle, a contradiction. Theorem 2. This completes the proof of theorem 1.

An example which shows that the condition that each vertex has out-degree at least two in theorem 1 is indispensable can be constructed from the 5-tournament $T_5$ with an edge $(x, y)$ such that $x \notin V(T_5)$ and $y \in V(T_5)$.

If $H$ contains a path $P_4$, $x$ cannot be an interior vertex of $P_4$ since $d(x) = 1$; furthermore it cannot be an end of $P_4$ since $d^-(x) = 0$. Thus $P_4 \subseteq T_5$ which contradicts lemma 1.

We conclude this paper by asking the following question:

Does there exist a 5-chromatic oriented graph which contains neither a 5-tournament nor $P_4$?
References


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